Lecture 18
11/18

Def A RR is an eqn that recursively defines a seq or array. Each term defined as a function of preceding terms.

A linear homogeneous RR of deg k w/ const coeffs is an RR of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} \quad (1)$$
$$= \sum_{i=1}^{k} c_i a_{n-i}$$

A seq satisfying (1) uniquely determined by the RR and the k initial conditions

$$a_0 = C_0, \ a_1 = C_1, \ \ldots, \ a_{k-1} = C_{k-1}$$

Ex $P_n = 1.2 P_{n-1}$ deg 1

$f_n = f_{n-1} + f_{n-2}$ deg 2

$a_n = a_{n-5}$ deg 5

$a_n = a_{n-1} + a_{n-2}^t$ nonlinear

$H_n = 2H_{n-1} + 1$ non homogeneous

$B_n = n B_{n-1}$ non const coeffs.

Thm Let $c_1, c_2 \in \mathbb{R}$. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2. Then seq $\{a_n\}$ is a sol to the RR $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \neq 0, 1, 2$ where α_1, α_2 consts.

Ex $a_n = a_{n-1} + 2a_{n-2}$ $a_0 = 2$

$r^2 - r - 2 = 0$ $a_1 = 7$

$(r-2)(r+1) = 0 \Rightarrow r = 2, -1$

$a_n = (-1)(2)^n + 3(-1)^n$

$a_n = \alpha_1(2)^n + \alpha_2(-1)^n$

$a_0 = \alpha_1 + \alpha_2 = 2$

$\alpha_1 = \alpha_1 \cdot 2 + \alpha_2 \cdot -1 = 7$

$3\alpha_1 = 9 \Rightarrow \alpha_1 = 3 \Rightarrow \alpha_2 = -1$

Ex $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$

$r^3 + r^2 - 4r - 4 = 0$ $a_0 = 8$, $a_1 = 6$, $a_2 = 26$

$r^2(r+1) - 4(r+1) = 0$

$(r^2-4)(r+1) \Rightarrow (r-2)(r+2)(r+1) = 0$

$r = 2, -2, -1$

$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$

$a_1 = \alpha_1(2) + (-2)\alpha_2 - 1\alpha_3 = 6$

$a_2 = \alpha_1(4) + 4\alpha_2 + \alpha_3 = 26$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -2 & -1 & 6 \\ 4 & 4 & 1 & 26 \end{array}\right]$$

~~R1 R2 → R2~~

$2R_1 - R_2 \rightarrow R_2$

$0\ 4\ 3 \,|\, 10$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 4 & 3 & 10 \\ 4 & 4 & 1 & 26 \end{array}\right]$$

$4R_1 - R_3 \rightarrow R_3$

$0\ 0\ 3 \,|\, 6$

$3c_3 = 6 \Rightarrow c_3 = 2$

$4c_2 + 6 = 10$ $c_2 = 1$

$c_1 + 1 + 2 = 8$

$c_1 = 5$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 4 & 3 & 10 \\ 0 & 0 & 3 & 16 \end{array}\right]$$

Thm Let $c_1, c_2 \in \mathbb{R}$ with $c_2 \neq 0$. Suppose $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A seq $\{a_n\}$ is a sol to the RR $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2,$ α_1, α_2 consts

Ex $a_n = 6a_{n-1} - 9a_{n-2}$ $\quad a_0 = 1$ $\quad a_1 = 6$ $\quad \cancel{a_0 = a_1 = 1}$

$\cancel{a_n}$ $\quad r^2 - 6r + 9 = 0$ $\quad \alpha_1 = 1 \quad \alpha_2 = 1$

$(r-3)(r-3) = 0$ $\quad a_n = \alpha_1 3^n + \alpha_2 3^n \cdot n$

$r = 3$

~~Thm If r_0 is a sol to the char eqn of multiplicity at least m+1~~ ~~$n^m r_0^n$ is a sol to the recurrence~~

Thm
Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Sps $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has k distinct roots $r_1, \ldots, r_k$. Then seq $\{a_n\}$ is a sol

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad \text{iff} \quad (1)$$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots \alpha_k r_k^n \quad n = 0, 1, 2, \quad \alpha_1, \ldots, \alpha_k \text{ consts}$$

Thm. Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Sps. $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has t distinct roots $r_1, \ldots, r_t$ w/ multiplicities $m_1, \ldots, m_t$. so $m_i \geq 1$ $i = 1, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then $\{a_n\}$ a sol to (1) iff

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

$n = 0, 1, 2,$ $\alpha_{i,j}$ consts $1 \leq i \leq t$ $\quad 0 \leq j \leq m_i - 1$

Ex 2, 2, 2, 5, 5, 9 are roots

- 3 roots w/ multiplicity 2
- 2 " " " 5
- 1 " " " 1

Sol form $(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2) 2^n + (\alpha_{2,0} + \alpha_{2,1} n) 5^n + \alpha_{3,0} 9^n$

+ .

Ex

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

$a_0 = 1$

$a_1 = -2$

$a_2 = -1$

$r^3 + 3\cancel{r}n^2 + 3r + 1 = 0$

$(r+1)^3 = 0$

$r = -1$

$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n$

$\alpha_{1,0} = 1$

$\alpha_{1,2} = -2$

$a_n = (1 + 3n - 2n^2)(-1)^n$

$\cancel{a_n = (1 + 3n + 3n^2}$

$a_0 = 1 = \alpha_{1,0}$

$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$

$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$

$\Rightarrow -2 = -1 - \alpha_{11} - \alpha_{12} \Rightarrow$

$-1 = 1 + 2\alpha_{11} + 4\alpha_{12}$

$\begin{matrix} -1 & -\alpha_{11} & -\alpha_{12} \\ -2 & 2\alpha_{11} & 4\alpha_{12} \end{matrix}$

$\left[\begin{array}{c|cc} -1 & -1 & -1 \\ -2 & 2 & 4 \end{array}\right]$ $2R_1 + R_2 \to R_2$ $\left[\begin{array}{c|cc} -1 & -1 & -1 \\ -4 & 0 & 2 \end{array}\right]$ $\Rightarrow 2\alpha_{1,2} = -4$

$\alpha_{1,2} = -2$

$\cancel{-N = -\alpha_{11} - (-2)} \Rightarrow \cancel{+B} = -\alpha_{11}$ $\alpha_{11} = 3$ ✓

$-2 = -1 - \alpha_{11} - (-2) \Rightarrow \cancel{-2} -1 = -\alpha_{11} + 2$

Linear Nonhomogeneous RR

Ex $a_n = 3a_{n-1} + 2n$

Def A L.NH. RR w/ const coef is of the form

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} + F(n)$$

$C_1, \dots, C_k \in \mathbb{R}$, $F(n)$ is a nonzero fn of n

The RR underlined is ~~an~~ called the associated homogeneous RR

Ex. $a_n = a_{n-1} + 2^n$

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

Remark Every sol to LN-HRR (w/ const coeffs) is the sum of a ~~Key~~ sol and the sol to the associated homogeneous RR

Thm If $\{a_n^{(P)}\}$ is a sol of the NHRR w/ constant coeffs

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} + F(n)$$

then every sol is of the form $\{a_n^{(P)} + a_n^{(h)}\}$

where $\{a_n^{(h)}\}$ is the sol of the associated homogeneous RR

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$$

if b/c $\{a_n^{(p)}\}$ is a sol of NHRR

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \dots + c_k a_{n-k}^{(p)} + F(n) \quad i$$

Sps $\{b_n\}$ is a second sol

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} + F(n) \quad ii$$

Subtract (i) from (ii)

$$b_n - a_n^{(p)} = c_1\left(b_{n-1} - a_{n-1}^{(p)}\right) + \dots + c_k\left(b_{n-k} - a_{n-k}^{(p)}\right)$$

$\Rightarrow \{b_n - a_n^{(p)}\}$ is a sol, call it $\{a_n^{(h)}\}$

$\Rightarrow b_n = a_n^{(p)} + a_n^{(h)} \ \forall n.$

$\square$

Ex $a_n = 3a_{n-1} + 2n \quad a_1 = 3$

$a_n = a_n^p + a_n^h = -n - \frac{3}{2} + \alpha \cdot 3^n$

$a_1 = 3, n = 1 \Rightarrow 3 = -1 - \frac{3}{2} + 3\alpha$

$\alpha = 11/6$

$\Rightarrow a_n = 3a_{n-1} \Rightarrow a_n^{(h)} = \alpha(3)^n$

Since $F(n) = 2n$ is a polynomial of degree 1

try linear function $\Rightarrow p_n = cn + d \quad c, d$ consts

$\Rightarrow a_n = 3a_{n-1} + 2n \Rightarrow (cn+d) = 3(c(n-1)+d) + 2n \quad \to c = -1, \ d = -3/2$

$cn + d = 3cn - 3c + 3d + 2n$

$0 = 2cn + 2n - 3c + 2d \Rightarrow 0 = n(2c+2) + (2d - 3c)$

$cn+d$ a sol iff

$\Rightarrow (2c+2) = 0$ and $(2d-3c) = 0$

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r = 2, 3$

$a_n^h = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$

$F(n) = 7^n$ so try $a_n^{(p)} = C \cdot 7^n$ with C const.

$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$

Factor 7^{n-2}

$\Rightarrow C \cdot 7^2 = 5 \cdot C \cdot 7 - 6 \cdot C + 7^2$

$49C = 35C - 6C + 49$

$20C = 49 \quad C = 49/20$

$a_n^p = \frac{49}{20} \cdot 7^n$

$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + \left(\frac{49}{20}\right) 7^n$

<u>Thm</u> Sps $\{a_n\}$ satisfying linear NHRR $a_n = C_1 a_{n-1} + \ldots + C_k a_{n-k} + F(n)$
$= \sum_{i=1}^{k} C_i a_{n-i} + F(n)$

with $C_1, \ldots, C_k$ ~~consts~~ $\in \mathbb{R}$, $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$

$b_0, \ldots b_t \in \mathbb{R}$ $= \left(\sum_{i=0}^{t} b_i n^{t-i}\right) s^n$

when s is not a root of char eqn of associated linear homo RR, $\exists$ a sol of the form $(P_t n^t + P_{t-1} n^{t-1} + \ldots + P_1 n + P_0) s^n$. when s is root w/ multiplicity m $\exists$ sol
$\Rightarrow n^m (P_t n^t + P_{t-1} n^{t-1} + \ldots P_1 n + P_0) s^n$